Qualche applicazione dell'IA in Matematica

Soglie per sequenze di random variables: definizione di λ_n

Definition

Dato $A \subseteq [0,1]^n$ definiamo

$$\Phi(A) = \int_{[0,1]^{n+1}} \mathbf{1}_A(x_1, \ldots, x_n)\big(1 - \mathbf{1}_A(x_2, \ldots, x_{n+1})\big)\, dx$$

e

$$\lambda_n := \sup_{A \subseteq [0,1]^n} \Phi(A).$$

Problema: calcolare λ_n.

Risultati noti

Appare in due lavori su sequenze di random variables (Trotter-Winkler 1998, Majer-Novaga 2012).

Questi i risultati noti:

- $\lambda_n \leq \lambda_{n+1}$,
- $\lambda_1 = 1/4$, $n/(2n+2) \leq \lambda_n \leq n/(2n+1)$ (Trotter-Winkler),
- $\lambda_n = n/(2n+2)$ se n è pari (Majer-Novaga).

In particolare, per $n = 3$ si ottiene $3/8 \leq \lambda_3 \leq \lambda_4 = 2/5$.

Congettura (Trotter-Winkler): $\lambda_3 = 3/8$.

Problema discreto

Per $q \in \mathbb{N}$, sia $[q] = \{0, 1, \ldots, q-1\}$. Per $S \subseteq [q]^3$:

$$N_q(S) = \#\{(a,b,c,d) \in [q]^4 : (a,b,c) \in S,\ (b,c,d) \notin S\},$$

$$\Lambda_q(S) = \frac{N_q(S)}{q^4}.$$

Il legame con il continuo avviene tramite partizione in cubetti q-adici:

$$\Phi(A_S) = \Lambda_q(S), \qquad A_S = \bigcup_{(a,b,c)\in S} \left[\frac{a}{q}, \frac{a+1}{q}\right) \times \ldots$$

Il problema si riduce a studiare $\sup_S \Lambda_q(S)$ e poi passare al limite per $q \to \infty$.

ChatGPT: disuguaglianza del 7–ciclo

Lemma

Per ogni $y = (y_i)_{i \in \mathbb{Z}/7\mathbb{Z}} \in \{0,1\}^7$, *si ha*

$$\sum_{i \in \mathbb{Z}/7\mathbb{Z}} y_i(1 - y_{i+1}) \leq 1 + \sum_{i \in \mathbb{Z}/7\mathbb{Z}} y_i(1 - y_{i+3}).$$

Upper bound discreto

Somma sulle 7-uple: Sommiamo ora la disuguaglianza del Lemma su tutte le sequenze $x = (x_0, \ldots, x_6) \in [q]^7$. Dato $S \subset [q]^3$ e ponendo $y_i(x) := 1_S(x_i, x_{i+1}, x_{i+2})$ per $x \in [q]^7$, abbiamo

$$\sum_{x\in[q]^7} \sum_i y_i(x)(1 - y_{i+1}(x)) \leq \sum_{x\in[q]^7} 1 + \sum_{x\in[q]^7} \sum_i y_i(x)(1 - y_{i+3}(x))$$

Lato Sinistro (Adiacenze): Il termine $y_i(x)(1 - y_{i+1}(x))$ dipende solo da 4 coordinate $(x_i, \ldots, x_{i+3})$. Per definizione, ci sono $N_q(S)$ modi di scegliere queste 4 in modo che la prima terna sia in S e la seconda no. Le altre 3 coordinate della 7-upla sono libere (fattore q^3).
Sommando su tutti i 7 indici i, otteniamo $7q^3 N_q(S)$.

Upper bound discreto

Lato Destro (Salti): Il termine $y_i(x)(1 - y_{i+3}(x))$ valuta due terne disgiunte: (x_i, x_{i+1}, x_{i+2}) e $(x_{i+3}, x_{i+4}, x_{i+5})$. Ci sono $|S|$ modi per scegliere la prima terna in S e $(q^3 - |S|)$ modi per scegliere la seconda fuori da S. L'unica coordinata rimanente (x_{i+6}) fornisce un fattore q.

Semplificazione: Il termine costante sommato su $[q]^7$ dà q^7. La disuguaglianza diventa:

$$7q^3 N_q(S) \leq q^7 + 7q|S|(q^3 - |S|)$$

Conclusione: Dividendo tutto per $7q^7$, abbiamo $\Lambda_q(S) \leq \frac{1}{7} + \alpha_q(S)(1 - \alpha_q(S)) \leq 1/4 + 1/7 = 11/28$.

Upper bound

Passando al limite $q \to +\infty$ otteniamo

Theorem

$$\lambda_3 \leq \frac{11}{28} < \frac{2}{5}.$$

Migliora la stima $\lambda_3 \leq \lambda_4 = \frac{2}{5}$ ottenuta da Majer-Novaga.

ChatGPT: l'ipotesi H_q

Dato $S \subset [q]^3$ definiamo

- $I_S(u,v) = |\{x : (x,u,v) \in S\}|$ (numero di entrate)
- $O_S(u,v) = |\{y : (u,v,y) \in S\}|$ (numero di uscite).

Diciamo che S soddisfa l'ipotesi H_q se esistono due "siti" speciali $a, b \in [q]$ dove valgono alcuni vincoli rigidi (ad es. $I_S(a,a) + O_S(a,a) = q$).

Raddoppio

Come passiamo da una configurazione a scala q a una a scala $2q$?

1. **Raddoppio diadico:** Prendiamo il lift T di S in $[2q]^3$.
2. **Perturbazione locale:** Modifichiamo chirurgicamente pochissimi punti vicino alle coordinate speciali (aggiungiamo 3 cubetti e ne togliamo 4).

Lemma

S soddisfa H_q, allora T soddisfa H_{2q} e

$$N_{2q}(T) = 16N_q(S) + 1.$$

Iterazione

Per far partire l'induzione, serve una configurazione iniziale che verifichi l'ipotesi.

Assemblando blocchi più piccoli in una struttura a strati, si riesce a definire esplicitamente un insieme $S_{16} \subset [16]^3$.

Si verifica che S_{16} soddisfa H_{16} con $a = 8$ e $b = 14$ e che

$$N_{16}(S_{16}) = 24849.$$

Lower bound e conclusione

Iterando il Lemma partendo da S_{16}, dalla ricorrenza $N_{2^{m+1}} = 16N_{2^m} + 1$ si ottiene

$$N_{2^m}(S_{2^m}) = \frac{91}{240}2^{4m} - \frac{1}{15}.$$

Passando al continuo ($m \to \infty$) otteniamo la densità limite:

Theorem

$$\lambda_3 \geq \frac{91}{240} > \frac{3}{8}$$

Questo risultato migliora il limite inferiore e smentisce la congettura originale di Trotter-Winkler ($\lambda_3 = 3/8$).

Attenzione agli errori

Talvolta vengono introdotti errori, anche banali. non facili da identificare. Se sono scritti in maniera convincente, il chatbot tende a non ricontrollarli.

Per questo è essere utile far verificare quanto prodotto anche ad altri programmi.

Esempio di errore

EXAMPLE OF WRONG LEMMA

For $u > 0$, let

$$F(u,1,1,u,1,u) = \frac{2\,J_{\mathcal{P}}(u)}{P(u)^{5/6}},$$

where

$$P(u) := u^3 + 7u^2 + 7u + 1,$$

and

$$J_{\mathcal{P}}(u) := u^2\sqrt{u+3} + 2u(u+2)^{3/2} + u\sqrt{2u+2} + 2(2u+1)^{3/2} + \sqrt{3u+1}.$$

Theorem 0.1. *For every $p, q > 0$,*

$$F(u,1,1,u,1,u) \;\geq\; F(1,1,1,1,1,1),$$

with equality if and only if $u = 1$.

Define the one–variable function

$$\widetilde{F}_{\mathcal{P}}(u) := F(u,1,1,u,1,u) = \frac{2\,J_{\mathcal{P}}(u)}{P(u)^{5/6}}.$$

As in the previous one-variable analyses of the $(1,5)$ and $(2,4)$ strata, the sign of the derivative is captured by the auxiliary function

$$\Psi_{\mathcal{P}}(u) := 6P(u)\,J'_{\mathcal{P}}(u) - 5P'(u)\,J_{\mathcal{P}}(u),$$

since

$$\widetilde{F}'_{\mathcal{P}}(u) = \frac{\Psi_{\mathcal{P}}(u)}{3\,P(u)^{11/6}} \qquad \Longrightarrow \qquad \operatorname{sgn}\,\widetilde{F}'_{\mathcal{P}}(u) = \operatorname{sgn}\,\Psi_{\mathcal{P}}(u).$$

Lemma 0.2. *Let $P(u)$, $J_{\mathcal{P}}(u)$ as above. Then, for every $u > 0$,*

$$\Psi'''_{\mathcal{P}}(u) = 6P(u)\,J^{(4)}_{\mathcal{P}}(u) + 13P'(u)\,J^{(3)}_{\mathcal{P}}(u) + 3P''(u)\,J''_{\mathcal{P}}(u) - 9P^{(3)}(u)\,J'_{\mathcal{P}}(u).$$

Esempio di errore

Lemma 0.3 (Closed form and positivity of $\Psi'''_{\mathcal{P}}$)**.** *For every $u > 0$, letting $P(u) = u^3 + 7u^2 + 7u + 1$, one has*

$$\begin{aligned}\Psi'''_{\mathcal{P}}(u) = {} & \underbrace{\left[-162u\sqrt{u+2} - 72u\sqrt{u+3} + 36\sqrt{u+2} + 84\sqrt{u+3} - 324\sqrt{2u+1} - 54\sqrt{2u+2}\right]}_{=:\mathcal{A}(u)} \\ & \underbrace{\left[\frac{9(45u^2+196u+91)}{2\sqrt{u+2}} + \frac{21(6u^2+30u+13)}{\sqrt{u+3}} + \frac{\sqrt{2}(42-9u)}{\sqrt{u+1}} - \frac{81}{\sqrt{3u+1}}\right]}_{=:\mathcal{B}(u)} \\ & \underbrace{\left[\frac{108P(u)}{(2u+1)^{5/2}} + \frac{36(3u+7)}{\sqrt{2u+1}} - \frac{78(3u^2+14u+7)}{(2u+1)^{3/2}}\right]}_{=:\mathcal{C}(u)} \\ & \underbrace{\left[\frac{1053(3u^2+14u+7)}{8(3u+1)^{5/2}} - \frac{27(3u+7)}{2(3u+1)^{3/2}} - \frac{3645P(u)}{8(3u+1)^{7/2}}\right]}_{=:\mathcal{D}(u)} \\ & \underbrace{\left[\frac{27uP(u)}{4(u+2)^{5/2}} - \frac{3(63u^3+350u^2+259u+24)}{4(u+2)^{3/2}}\right]}_{=:\mathcal{E}_2(u)} \\ & \underbrace{\left[\frac{3u(87u^3+518u^2+427u+48)}{8(u+3)^{5/2}} - \frac{3(54u^3+273u^2+175u+12)}{2(u+3)^{3/2}} - \frac{45u^2P(u)}{8(u+3)^{7/2}}\right]}_{=:\mathcal{E}_3(u)} \\ & \underbrace{\left[\frac{3\sqrt{2}(63u^3+350u^2+259u+24)}{8(u+1)^{5/2}} - \frac{3\sqrt{2}(45u^2+196u+91)}{4(u+1)^{3/2}} - \frac{45\sqrt{2}\,u(u^2+6u+1)}{8(u+1)^{5/2}}\right]}_{=:\mathcal{E}_1(u)}. \end{aligned} \tag{$\Psi'''_{\mathcal{P}}$}$$

In particular, $\Psi'''_{\mathcal{P}}(u) > 0$ for all $u > 0$.

Proof. **Step 1: derivation of the closed form.** Starting from Lemma 0.2 we differentiate the five terms defining $J_{\mathcal{P}}(u)$:

$$u^2\sqrt{u+3}, \qquad 2u(u+2)^{3/2}, \qquad u\sqrt{2u+2} = \sqrt{2}\,u\sqrt{u+1},$$
$$2(2u+1)^{3/2}, \qquad \sqrt{3u+1}.$$

Each term has derivatives up to order 4 that are linear combinations of expressions of the form $\sqrt{u+c}$,$(u+c)^{-1/2}$,$(u+c)^{-3/2}$,$(u+c)^{-5/2}$,$(u+c)^{-7/2}$ with $c \in \{1,2,3\}$, and similarly for $(2u+1)$ and $(3u+1)$. Substituting those derivatives into the master identity of Lemma 0.2 and simplifying using $(u+1)^2 = u^2+2u+1$, $(u+2)^2 = u^2+4u+4$, $(u+3)^2 = u^2+6u+9$, $(2u+1)^2 = 4u^2+4u+1$, $(3u+1)^2 = 9u^2+6u+1$ produces the explicit decomposition ($\Psi'''_{\mathcal{P}}$). (The algebra is straightforward but long; it amounts to collecting like powers of $(u+1)$, $(u+2)$, $(u+3)$, $(2u+1)$ and $(3u+1)$.)

We prove $\Psi'''_{\mathcal{P}}(u) > 0$ for all $u > 0$ by splitting into two regimes:

$$0 < u \le 1, \qquad \text{and} \qquad u \ge 1.$$

Throughout we use only the following elementary facts: (i) if $H(u) = \frac{N(u)}{(u+c)^\alpha}$ with $c > 0$ and $\alpha > 0$, then

$$H'(u) = \frac{N'(u)(u+c) - \alpha N(u)}{(u+c)^{\alpha+1}};$$

Esempio di errore

which follow from squaring both sides.

(a) The range $0 < u \le 1$. We further split $0 < u \le 1$ into three subintervals

$$(0, \tfrac{1}{10}], \qquad [\tfrac{1}{10}, \tfrac{1}{5}], \qquad [\tfrac{1}{5}, 1].$$

The reason is that the negative square-root block $\mathcal{A}(u)$ becomes more negative with u, while the positive rational block $\mathcal{D}(u)$ becomes smaller with u.

Monotonicity facts used in all three subintervals. We record six monotonicity statements that follow by differentiating explicitly.

- The block $\mathcal{A}(u)$ is strictly decreasing on $(0, \infty)$: each summand of $\mathcal{A}$ has negative derivative, e.g.

$$\frac{d}{du}\left(-162u\sqrt{u+2}\right) = -162\sqrt{u+2} - \frac{81u}{\sqrt{u+2}} < 0,$$

  and similarly for the other five summands.
- The block $\mathcal{C}(u)$ is strictly increasing on $(0, \infty)$. Indeed, combining the three terms in $\mathcal{C}$ over the common denominator $(2u+1)^{5/2}$ gives

$$\mathcal{A}_3(u) = \frac{72u^3 - 222u^2 - 312u - 186}{(2u+1)^{5/2}} = -\frac{186 + 312u + 222u^2 - 72u^3}{(2u+1)^{5/2}}.$$

  Let $N(u) := 186 + 312u + 222u^2 - 72u^3$ and a direct computation yields $N'(u)(2u+1) - 5N(u) = 72u^3 + 438u^2 + 492u + 618 > 0$ for $u > 0$.
- The block $\mathcal{D}(u)$ is strictly decreasing on $(0, \infty)$, and is everywhere positive. Indeed, combining the three terms of $\mathcal{D}$ over the common denominator $8(3u+1)^{7/2}$ gives

$$\mathcal{D}(u) = \frac{2916u^3 + 13122u^2 + 6480u + 2970}{8(3u+1)^{7/2}},$$

  whose numerator has positive coefficients, so $\mathcal{D}(u) > 0$. Moreover,

$$\mathcal{D}'(u) = \frac{N_D'(u)(3u+1) - \frac{21}{2}N_D(u)}{8(3u+1)^{9/2}},$$

  with $N_D(u) = 2916u^3 + 13122u^2 + 6480u + 2970$, and

$$N_D'(u)(3u+1) - \frac{21}{2}N_D(u) = -4374u^3 - 50301u^2 - 22356u - 24705 < 0.$$

  Thus $\mathcal{D}'(u) < 0$.
- The blocks $\mathcal{E}_2(u)$ and $\mathcal{E}_3(u)$ are strictly decreasing on $(0, \infty)$ and strictly negative. Indeed, after combining to a single fraction one finds

$$\mathcal{E}_2(u) = \frac{-162u^4 - 1239u^3 - 2688u^2 - 1599u - 144}{4(u+2)^{5/2}},$$

$$\mathcal{E}_3(u) = \frac{-432u^5 - 5142u^4 - 21960u^3 - 38286u^2 - 19332u - 1296}{8(u+3)^{7/2}},$$

  so both are negative. Writing $\mathcal{E}_j(u) = -\frac{N_j(u)}{(u+c_j)^{\alpha_j}}$ with N_j having positive coefficients, one checks mechanically that

$$N_2'(u)(u+2) - \frac{5}{2}N_2(u) > 0, \qquad N_3'(u)(u+3) - \frac{7}{2}N_3(u) > 0 \qquad (u > 0),$$

  hence $\mathcal{E}_2'(u) < 0$ and $\mathcal{E}_3'(u) < 0$.
- The block $\mathcal{E}_1(u)$ is strictly increasing on $(0, 1]$. Indeed,

$$\mathcal{E}_1(u) = \frac{\sqrt{2}\,(-126u^3 - 666u^2 - 990u - 474)}{8(u+1)^{5/2}} = -\frac{\sqrt{2}\,N_1(u)}{8(u+1)^{5/2}},$$

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and the bracket expands to

$$N_1'(u)(u+1) - \frac{5}{2}N_1(u) = 63u^3 + 45u^2 - 153u - 195 < 0 \qquad (0 < u \le 1),$$

so $\mathcal{E}_1'(u) > 0$ on $(0, 1]$.

Subinterval $(0, \frac{1}{10}]$. On $0 < u \le \frac{1}{10}$ we use: $\mathcal{A}(u) \ge \mathcal{A}(\frac{1}{10})$ (decreasing), $\mathcal{C}(u) \ge \mathcal{C}(0)$ (increasing), $\mathcal{D}(u) \ge \mathcal{D}(\frac{1}{10})$ (decreasing), $\mathcal{E}_2(u) \ge \mathcal{E}_2(\frac{1}{10})$ and $\mathcal{E}_3(u) \ge \mathcal{E}_3(\frac{1}{10})$ (decreasing), and $\mathcal{E}_1(u) \ge \mathcal{E}_1(0)$ (increasing). We also lower bound the positive block $\mathcal{B}(u)$ by its value at $u = 0$, using only the inequalities $\sqrt{u+2} \le \sqrt{2}$, $\sqrt{u+3} \le \sqrt{3}$, $\sqrt{u+1} \le 1$ and $\sqrt{3u+1} \ge 1$ for $0 < u \le \frac{1}{10}$:

$$\mathcal{B}(u) \ge \mathcal{B}(0) = \frac{819}{2\sqrt{2}} + \frac{273}{\sqrt{3}} + 42\sqrt{2} - 81.$$

A direct substitution gives the exact endpoint values

$$\mathcal{A}\Big(\frac{1}{10}\Big) = -\frac{324}{5}\sqrt{30} - \frac{54}{5}\sqrt{55} + \frac{99}{50}\sqrt{210} + \frac{192}{25}\sqrt{310},$$
$$\mathcal{C}(0) = -186, \qquad \mathcal{D}\Big(\frac{1}{10}\Big) = \frac{469017}{28561}\sqrt{130},$$
$$\mathcal{E}_2\Big(\frac{1}{10}\Big) = -\frac{1098}{1225}\sqrt{210}, \qquad \mathcal{E}_3\Big(\frac{1}{10}\Big) = -\frac{90863463}{184704200}\sqrt{310},$$
$$\mathcal{E}_1(0) = -\frac{237}{4}\sqrt{2}.$$

Therefore for $0 < u \le \frac{1}{10}$,

$$\Psi_{\mathcal{P}}'''(u) \ge \mathcal{A}\Big(\frac{1}{10}\Big) + \mathcal{B}(0) + \mathcal{C}(0) + \mathcal{D}\Big(\frac{1}{10}\Big) + \mathcal{E}_2\Big(\frac{1}{10}\Big) + \mathcal{E}_3\Big(\frac{1}{10}\Big) + \mathcal{E}_1(0).$$

Bounding each radical with the rational inequalities listed at the start of Step 2 (always in the direction that makes the right-hand side smaller), one obtains the fully explicit lower bound

$$\Psi_{\mathcal{P}}'''(u) \ge \frac{28983914554167493}{646228740384500} > \frac{1}{2} \qquad \Big(0 < u \le \frac{1}{10}\Big).$$

Every term is now rational, and after clearing denominators we obtain the stated purely rational lower bound.

Subinterval $[\frac{1}{10}, \frac{1}{5}]$. On $\frac{1}{10} \le u \le \frac{1}{5}$ we use

$$\mathcal{A}(u) \ge \mathcal{A}\Big(\frac{1}{5}\Big), \quad \mathcal{B}(u) \ge \mathcal{B}\Big(\frac{1}{10}\Big), \quad \mathcal{C}(u) \ge \mathcal{C}\Big(\frac{1}{10}\Big), \quad \mathcal{D}(u) \ge \mathcal{D}\Big(\frac{1}{5}\Big),$$
$$\mathcal{E}_2(u) \ge \mathcal{E}_2\Big(\frac{1}{5}\Big), \quad \mathcal{E}_3(u) \ge \mathcal{E}_3\Big(\frac{1}{5}\Big), \quad \mathcal{E}_1(u) \ge \mathcal{E}_1\Big(\frac{1}{10}\Big),$$

using the monotonicity statements above (note that $\mathcal{E}_1$ is increasing on $(0, 1]$). Substituting the endpoints and bounding radicals again by rational numbers (as in the previous subinterval) yields

$$\Psi_{\mathcal{P}}'''(u) \ge \frac{2500086479368551}{76897300480000} > 1 \qquad \Big(\frac{1}{10} \le u \le \frac{1}{5}\Big).$$

Subinterval $[\frac{1}{5}, 1]$ (zero–salti). We now push the last part of the proof to a completely local, interval–wise, "mechanical" estimate. We subdivide

$$\Big[\frac{1}{5}, 1\Big] = \Big[\frac{1}{5}, \frac{1}{2}\Big] \cup \Big[\frac{1}{2}, \frac{3}{4}\Big] \cup \Big[\frac{3}{4}, 1\Big].$$

On each subinterval we use only: (i) the monotonicity facts recorded above; (ii) exact endpoint evaluations (kept symbolically, with radicals); (iii) explicit rational bounds for each radical, obtained by squaring.

Esempio di errore

$$\mathcal{E}_2(u) \ge \mathcal{E}_2\Big(\frac{1}{2}\Big), \qquad \mathcal{E}_3(u) \ge \mathcal{E}_3\Big(\frac{1}{2}\Big), \qquad \mathcal{E}_1(u) \ge \mathcal{E}_1\Big(\frac{1}{5}\Big),$$

and for $\mathcal{B}$ we use the point $u = \frac{1}{5}$ (since every denominator in $\mathcal{B}$ is increasing and the only potentially negative numerator $\sqrt{2}(42 - 9u)$ is still positive for $u \le \frac{1}{2}$):

$$\mathcal{B}(u) \ge \mathcal{B}\Big(\frac{1}{5}\Big) \qquad \Big(\frac{1}{5} \le u \le \frac{1}{2}\Big).$$

Hence

$$\Psi'''_{\mathcal{P}}(u) \ge \mathcal{A}\Big(\frac{1}{2}\Big) + \mathcal{B}\Big(\frac{1}{5}\Big) + \mathcal{C}\Big(\frac{1}{5}\Big) + \mathcal{D}\Big(\frac{1}{2}\Big) + \mathcal{E}_2\Big(\frac{1}{2}\Big) + \mathcal{E}_3\Big(\frac{1}{2}\Big) + \mathcal{E}_1\Big(\frac{1}{5}\Big). \qquad (\star_{1/2})$$

All seven endpoint quantities are explicit radicals; for completeness we list them (exactly as produced by substitution):

$$\mathcal{A}\Big(\frac{1}{2}\Big) = -81\sqrt{10} - 36\sqrt{14} + \frac{9}{2}\sqrt{70} + \frac{21}{2}\sqrt{154},$$

$$\mathcal{C}\Big(\frac{1}{5}\Big) = -\frac{408}{125}\sqrt{15}, \qquad \mathcal{D}\Big(\frac{1}{2}\Big) = \frac{N_D(\frac{1}{2})}{8(\frac{5}{2})^{7/2}} = \frac{(2916 \cdot \frac{1}{8} + 13122 \cdot \frac{1}{4} + 6480 \cdot \frac{1}{2} + 2970)}{8(\frac{5}{2})^{7/2}},$$

$$\mathcal{E}_2\Big(\frac{1}{2}\Big) = -\frac{N_2(\frac{1}{2})}{4(\frac{5}{2})^{5/2}}, \qquad \mathcal{E}_3\Big(\frac{1}{2}\Big) = -\frac{N_3(\frac{1}{2})}{8(\frac{7}{2})^{7/2}}, \qquad \mathcal{E}_1\Big(\frac{1}{5}\Big) = -\frac{\sqrt{2}\,N_1(\frac{1}{5})}{8(\frac{6}{5})^{5/2}},$$

and $\mathcal{B}(\frac{1}{5})$ is expanded explicitly by substituting $u = \frac{1}{5}$ into the defining formula for $\mathcal{B}(u)$ in $(\Psi'''_{\mathcal{P}})$ and collecting like radicals. (We list below all radicals that appear at the endpoints and the rational bounds used for each.) Now we bound every radical appearing in $(\star_{1/2})$ by a rational number, always in the direction that makes the right-hand side smaller. The following list is sufficient for this subinterval (all verified by squaring):

$$\sqrt{10} > \frac{31}{10}, \quad \sqrt{14} > \frac{37}{10}, \quad \sqrt{15} > \frac{387}{100}, \quad \sqrt{70} > \frac{83}{10}, \quad \sqrt{154} > \frac{124}{10},$$

$$\sqrt{2} > \frac{7}{5}, \quad \sqrt{3} > \frac{173}{100}, \quad \sqrt{\frac{5}{2}} > \frac{158}{100}, \quad \sqrt{\frac{7}{2}} > \frac{187}{100}, \quad \sqrt{\frac{6}{5}} > \frac{109}{100}.$$

Substituting these bounds into $(\star_{1/2})$ and clearing denominators gives a purely rational lower bound:

$$\Psi'''_{\mathcal{P}}(u) \ge \frac{1097343881609251}{21146726400000} > 1 \qquad \Big(\frac{1}{5} \le u \le \frac{1}{2}\Big).$$

Subinterval $[\frac{1}{2}, \frac{3}{4}]$. On $\frac{1}{2} \le u \le \frac{3}{4}$ we use

$$\mathcal{A}(u) \ge \mathcal{A}\Big(\frac{3}{4}\Big), \quad \mathcal{B}(u) \ge \mathcal{B}\Big(\frac{1}{2}\Big), \quad \mathcal{C}(u) \ge \mathcal{C}\Big(\frac{1}{2}\Big), \quad \mathcal{D}(u) \ge \mathcal{D}\Big(\frac{3}{4}\Big),$$

$$\mathcal{E}_2(u) \ge \mathcal{E}_2\Big(\frac{3}{4}\Big), \quad \mathcal{E}_3(u) \ge \mathcal{E}_3\Big(\frac{3}{4}\Big), \quad \mathcal{E}_1(u) \ge \mathcal{E}_1\Big(\frac{1}{2}\Big),$$

again by monotonicity (note that $\mathcal{E}_1$ is increasing). Thus $\Psi'''_{\mathcal{P}}(u)$ is bounded from below by the sum of the seven endpoint values at $u = \frac{3}{4}$ (or as specified above). Expanding these endpoint values gives a finite linear combination of radicals

$$\sqrt{\frac{7}{4}}, \sqrt{\frac{11}{4}}, \sqrt{\frac{15}{4}}, \sqrt{\frac{5}{2}}, \sqrt{\frac{9}{2}}, \sqrt{\frac{5}{3}}, \sqrt{\frac{13}{4}}, \ldots$$

(listed explicitly below). We bound *each* radical by an explicit rational number by squaring; for instance

$$\sqrt{\frac{11}{4}} > \frac{331}{200}, \quad \sqrt{\frac{15}{4}} > \frac{387}{200}, \quad \sqrt{\frac{9}{2}} > \frac{212}{100}, \quad \sqrt{\frac{13}{4}} > \frac{180}{100}.$$

Esempio di errore

Subinterval $[\frac{3}{4}, 1]$. On $\frac{3}{4} \le u \le 1$ we use

$$\mathcal{A}(u) \ge \mathcal{A}(1), \quad \mathcal{B}(u) \ge \mathcal{B}\Big(\frac{3}{4}\Big), \quad \mathcal{C}(u) \ge \mathcal{C}\Big(\frac{3}{4}\Big), \quad \mathcal{D}(u) \ge \mathcal{D}(1),$$

$$\mathcal{E}_2(u) \ge \mathcal{E}_2(1), \quad \mathcal{E}_3(u) \ge \mathcal{E}_3(1), \quad \mathcal{E}_1(u) \ge \mathcal{E}_1\Big(\frac{3}{4}\Big),$$

again by monotonicity. Using the already–listed exact values

$$\mathcal{A}(1) = -450\sqrt{3} - 84, \qquad \mathcal{D}(1) = \frac{1593}{64}, \qquad \mathcal{E}_2(1) = -54\sqrt{3}, \qquad \mathcal{E}_3(1) = -\frac{5403}{64},$$

together with the explicit radical expansions of $\mathcal{B}(\frac{3}{4})$, $\mathcal{C}(\frac{3}{4})$ and $\mathcal{E}_1(\frac{3}{4})$ obtained by direct substitution into $(\Psi'''_{\mathcal{P}})$, and bounding *each* radical by a rational number via squaring as listed below.

Complete list of radical bounds used on $[\frac{3}{4}, 1]$. After expanding the endpoint sum (with $u = \frac{3}{4}$ and $u = 1$ as specified above), the only radicals that occur are among

$$\sqrt{2},\ \sqrt{3},\ \sqrt{4},\ \sqrt{\frac{7}{4}},\ \sqrt{\frac{5}{2}},\ \sqrt{\frac{7}{2}},\ \sqrt{\frac{11}{4}},\ \sqrt{\frac{13}{4}},\ \sqrt{\frac{15}{4}}.$$

We bound each of them by rational numbers (choosing lower or upper bounds depending on the sign of its coefficient). Every inequality below is checked by squaring.

$$\begin{aligned}
\sqrt{2} &> \frac{7}{5}, & \sqrt{2} &< \frac{71}{50},\\
\sqrt{3} &> \frac{173}{100}, & \sqrt{3} &< \frac{174}{100},\\
&& \sqrt{4} = 2,\\
\sqrt{\frac{7}{4}} &> \frac{132}{100}, & \sqrt{\frac{7}{4}} &< \frac{133}{100},\\
\sqrt{\frac{5}{2}} &> \frac{79}{50}, & \sqrt{\frac{5}{2}} &< \frac{159}{100},\\
\sqrt{\frac{7}{2}} &> \frac{187}{100}, & \sqrt{\frac{7}{2}} &< \frac{188}{100},\\
\sqrt{\frac{11}{4}} &> \frac{165}{100}, & \sqrt{\frac{11}{4}} &< \frac{167}{100},\\
\sqrt{\frac{13}{4}} &> \frac{18}{10}, & \sqrt{\frac{13}{4}} &< \frac{181}{100},\\
\sqrt{\frac{15}{4}} &> \frac{193}{100}, & \sqrt{\frac{15}{4}} &< \frac{194}{100}.
\end{aligned}$$

Substituting these bounds into the expanded endpoint sum and clearing denominators yields the stated purely rational lower bound.

We obtain the fully rational inequality

$$\Psi'''_{\mathcal{P}}(u) \ge \frac{401567331}{384000000} > 1 \qquad \Big(\frac{3}{4} \le u \le 1\Big).$$

Combining the three subintervals gives $\Psi'''_{\mathcal{P}}(u) > 0$ for all $\frac{1}{5} \le u \le 1$. Together with the already treated subintervals $(0, \frac{1}{10}]$ and $[\frac{1}{10}, \frac{1}{5}]$, this concludes $\Psi'''_{\mathcal{P}}(u) > 0$ for all $0 < u \le 1$.

(b) The range $u \ge 1$. For $u \ge 1$ we bound every shifted factor by a constant multiple of u:

Esempio di errore

We apply these inequalities term-by-term to $(\Psi_{\mathcal{P}}''')$, always replacing a *positive* term by a smaller quantity and a *negative* term by a more negative quantity, thereby producing a lower bound. After rewriting each summand as $c\,u^{3/2}$ times a numerical factor, one obtains

$$\Psi_{\mathcal{P}}'''(u) \geq \left(\frac{45}{8} - \frac{9}{\sqrt{2}}\right) u^{3/2} \qquad (u \geq 1).$$

To make this completely explicit: the coefficient is positive because $\sqrt{2} > \frac{7}{5}$ implies $\frac{9}{\sqrt{2}} < \frac{45}{7}$ and hence

$$\frac{45}{8} - \frac{9}{\sqrt{2}} > \frac{45}{8} - \frac{45}{7} = \frac{45}{56} > 0.$$

Thus $\Psi_{\mathcal{P}}'''(u) > 0$ for all $u \geq 1$.

Combining (a) and (b) shows $\Psi_{\mathcal{P}}'''(u) > 0$ for every $u > 0$. $\square$

By Lemma 0.3, $\Psi_{\mathcal{P}}''$ is strictly increasing on $(0, \infty)$ and hence $\Psi_{\mathcal{P}}$ is strictly convex. Since $\Psi_{\mathcal{P}}(1) = 0$ and $\Psi_{\mathcal{P}}(0^+) < 0 < \Psi_{\mathcal{P}}'(1)$, strict convexity forces $\Psi_{\mathcal{P}}(u) < 0$ for $0 < u < 1$ and $\Psi_{\mathcal{P}}(u) > 0$ for $u > 1$. Therefore $\widetilde{F}_{\mathcal{P}}$ has a unique global minimum at $u = 1$, proving Theorem 0.1.

Osservazioni e suggerimenti

- Più adatto a problemi di conteggio o computazionali che a identificare nuove strategie, anche se potrebbe trovare utili analogie in letteratura.
- Le dimostrazioni sono spesso "poco umane" e quindi difficili da verificare.
- Non controlla davvero i propri errori. Usare anche altri programmi per controlli incrociati.
- Se impiega troppo tempo tende a rispondere a caso. Dare sempre istruzioni chiare e vincolanti (prompt).
- Suddividere le richieste in passaggi successivi, in modo da procedere per gradi.

Esempio di prompt

Da ora in poi non confermare per default né assumere corrette le mie conclusioni. Agisci come partner critico, non come assistente accondiscendente.
Per ogni mia idea: Analizza le mie assunzioni: quali premesse implicite potrebbero essere false? Fornisci controargomentazioni: adotta lo sguardo di uno scettico informato; porta le obiezioni migliori e indica dati/condizioni a supporto.
Metti alla prova il mio ragionamento. Esegui un test di logica: dove si spezza la catena (premesse non provate, causalità dubbia, generalizzazioni, alternative escluse, fallacie)?
Offri prospettive alternative: Adotta la prospettiva di uno scettico informato: presenta le obiezioni migliori (non le più facili), indica in quali condizioni potrebbero avere ragione e quali dati le sosterrebbero.
Privilegia la correttezza alla compiacenza. Non devi darmi ragione necessariamente. Se fatti o logica mi smentiscono, correggimi e spiega con precisione dove/perché e cosa serve per renderla solida.
Separa fatti e opinioni: etichetta sempre [Fatti] / [Inferenze] / [Giudizi].
Stile: risposte approfondite, chiare, precise, puntuali; evita giri di parole, slogan, brevità eccessiva o ripetizioni.

Mantieni rigore costruttivo: non discutere per sport, ma guida verso chiarezza, accuratezza e onestà intellettuale. Se scivolo in bias di conferma, assunzioni non verificate o confondo fatti e opinioni, segnalalo subito. Affiniamo sia le conclusioni sia il processo con cui ci arriviamo.